

# Relativistic Time Transfer in the Solar System

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**Abstract**—Relativity has become an important aspect of modern precise timekeeping systems. Thus, far from being simply a textbook problem or merely of scientific interest, the analysis of relativistic effects on time measurement is an essential practical consideration. This paper outlines the fundamental concepts of relativistic time transfer and describes the details of the mathematical model. The magnitudes of various relativistic effects for clocks on GPS satellites, other Earth satellites, on Mars, or on the Moon are derived.

## I. INTRODUCTION

Relativity has become an important aspect of modern precise timekeeping systems. The Global Positioning System (GPS) is an example of an engineering system in which the recognition of appropriate relativistic effects are necessary for its successful operation. Similarly, relativistic transformations between clocks throughout the solar system will be required in future space missions. The purpose of this paper is to describe the theoretical principles for relativistic time transfer. The major relativistic effects are derived for time transfer between clocks on the Earth's surface to clocks on Earth-orbiting satellites, Mars, and the Moon.

## II. THE PRINCIPLE OF RELATIVITY

According to the special theory of relativity, formulated by Einstein in 1905, the laws of physics should have the same form in every inertial frame of reference. This postulate is known as the Principle of Relativity. Thus, in addition to the laws of mechanics, Maxwell's equations of electromagnetism should be valid in all inertial frames.

A fundamental prediction of Maxwell's equations is the existence of electromagnetic waves that propagate in vacuum at the speed of light  $c$ . Therefore, the speed of light must be the same in every inertial frame. In 1908 Minkowski recognized that this property could be expressed by the invariance of a four-dimensional space-time interval

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \quad (1)$$

such that the equation  $ds^2 = 0$  represents the expanding spherical wavefront of a light signal in any inertial frame. In addition,  $ds^2 = -c^2 d\tau^2$  represents the time  $\tau$  recorded by a clock at rest. The coordinate transformation that preserves the invariance of this expression is the Lorentz transformation.

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The theory of space, time, and gravitation according to the general theory of relativity is founded upon the notion that the Riemannian space-time interval

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \quad (2)$$
$$= g_{00} c^2 dt^2 + 2 \sum_{j=1}^3 g_{0j} c dt dx^j + \sum_{i=1}^3 \sum_{j=1}^3 g_{ij} dx^i dx^j$$

should be invariant under an arbitrary coordinate transformation [1]. This equation is the most general form of Eq. (1). The fundamental mathematical object is the metric tensor  $g_{\mu\nu}$ , whose components are symmetric in the indices  $\mu, \nu$  (that is,  $g_{\mu\nu} = g_{\nu\mu}$ ) and are functions of the coordinates  $x^\alpha \equiv (ct, x^i)$ . The metric tensor plays the role of the gravitational potentials.

## III. PROPER TIME AND COORDINATE TIME

In the theory of general relativity, there are two kinds of time. Proper time  $\tau$  is the actual reading of a clock. The proper times are different for clocks in different states of motion and in different gravitational potentials. The proper time measured by a clock may be compared to the proper time measured by another clock through the intermediate variable  $t$  called coordinate time, which, by definition, has the same value everywhere for a given event. Thus two events are simultaneous if their coordinate times are equal. The relationship between coordinate time and proper time depends on the velocity of the clock and the gravitational potential at the location of the clock. It is established through the invariance of the four-dimensional space-time interval.

For a transported clock, the space-time interval is

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \equiv -c^2 d\tau^2, \quad (3)$$

where  $\tau$  is the proper time recorded by the clock. For a given coordinate system, this equation establishes a well-defined transformation between coordinate time and proper time. The coordinate time is arbitrary, as ultimately the comparison is made between two proper times. For an electromagnetic signal, the space-time interval satisfies the condition

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu = 0. \quad (4)$$

In an inertial coordinate system with no gravitation, as in special relativity, the metric components are given by  $-g_{00} = 1$ ,  $g_{11} = g_{22} = g_{33} = 1$ , and  $g_{\mu\nu} = 0$  for  $\mu \neq \nu$ . Then for a transported clock,  $ds^2 = -c^2 (1 - v^2/c^2) dt^2 = -c^2 d\tau^2$  where  $v$  is the clock velocity. This implies the phenomenon of time dilation of a moving clock relative to a stationary clock. For a light signal,  $ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = 0$  as expressed by Eq. (1), whose invariance implies that the speed of light is  $c$  in all inertial systems.

#### IV. EARTH ORBITING SATELLITE CLOCKS

To a sufficient approximation in the analysis of clock transport, the components of the metric tensor in an Earth-Centered Inertial (ECI) coordinate system are given by  $-g_{00} \approx 1 - 2U/c^2$ ,  $g_{0j} = 0$ , and  $g_{ij} \approx \delta_{ij}$ , where  $U$  is the Newtonian gravitational potential and  $\delta_{ij}$  is the Kronecker delta. Thus by Eq. (3), for a clock onboard a satellite the elapsed coordinate time is given by the integral

$$\Delta t = \int_{\tau_0}^{\tau} \left( 1 + \frac{1}{c^2} U + \frac{1}{2} \frac{1}{c^2} v^2 \right) d\tau. \quad (5)$$

The first term under the integral is the elapsed proper time, the second term is the correction due to the gravitational potential  $U$  (gravitational redshift), and the third term is the correction due to the velocity  $v$  of the satellite (time dilation).

It is convenient to apply a change of scale to define a new coordinate time

$$\Delta t' = \left( 1 - \frac{1}{c^2} W_0 \right) \Delta t = \int_{\tau_0}^{\tau} \left\{ 1 + \frac{1}{c^2} (U - W_0) + \frac{1}{2} \frac{1}{c^2} v^2 \right\} d\tau, \quad (6)$$

where  $W_0 = 6.2637 \times 10^7 \text{ m}^2/\text{s}^2$  is the geopotential over the surface of the Earth, which is a constant. Then the coordinate time  $\Delta t'$  corresponds to the proper time registered by a clock at rest on the geoid and the clock becomes a coordinate clock. Upon integration, the elapsed coordinate time for an Earth-orbiting satellite clock is

$$\Delta t' = \left( 1 + \frac{3}{2} \frac{1}{c^2} \frac{GM}{a} - \frac{1}{c^2} W_0 \right) \Delta \tau + \frac{2}{c^2} \sqrt{GM a} e \sin E, \quad (7)$$

where  $a$  is the orbital semimajor axis,  $e$  is the orbital eccentricity, and  $E$  is the eccentric anomaly. The first term represents a constant rate offset between the satellite clock and a clock on the rotating geoid. The second term is a relativistic periodic correction due to the orbital eccentricity. This term may be expressed without approximation as

$$\Delta t_{rel} = \frac{2}{c^2} \sqrt{GM a} e \sin E = \frac{2 \mathbf{r} \cdot \mathbf{v}}{c^2}, \quad (8)$$

where  $\mathbf{r}$  and  $\mathbf{v}$  are the position and velocity vectors of the satellite. As  $\mathbf{r} \cdot \mathbf{v}$  is a scalar, it may be evaluated in either the ECI or ECEF coordinate system.

#### V. ELECTROMAGNETIC SIGNALS

By Eq. (4), the coordinate time of propagation of an electromagnetic signal is

$$\Delta t = \frac{\rho}{c} + \frac{1}{c} \int_{\text{path}} \sum_{j=1}^3 \frac{g_{0j}}{-g_{00}} dx^j, \quad (9)$$

where  $\rho$  is the propagation path length.

In an ECI frame the path length from transmitter to receiver is

$$\begin{aligned} \rho &= |\mathbf{r}_R(t_R) - \mathbf{r}_T(t_T)| \\ &\approx |\Delta \mathbf{r} + \mathbf{v}_R (t_R - t_T)| \approx |\Delta \mathbf{r}| + \frac{1}{c} \Delta \mathbf{r} \cdot \mathbf{v}_R, \end{aligned} \quad (10)$$

where  $\mathbf{r}_T(t_T)$  is the transmitter position at the coordinate time of transmission,  $\mathbf{r}_R(t_R)$  is the receiver position at the coordinate time of reception,  $\Delta \mathbf{r} = \mathbf{r}_R(t_R) - \mathbf{r}_T(t_T)$  is the transmitter and receiver separation at the coordinate time of transmission, and  $\mathbf{v}_R$  is the receiver velocity. The receiver “velocity correction” to the signal propagation time is

$$\Delta t_v = \Delta \mathbf{r} \cdot \mathbf{v}_R / c^2. \quad (11)$$

The integral term is called the Sagnac effect. In an ECI frame, this term is zero. In a rotating Earth-Centered Earth-Fixed (ECEF) coordinate system, the metric components are approximately  $-g_{00} \approx 1$ ,  $g_{0j} = (\boldsymbol{\omega} \times \mathbf{r})_j / c$ , and  $g_{ij} \approx \delta_{ij}$ , where  $\boldsymbol{\omega}$  is the Earth’s rotational angular velocity. Thus the Sagnac effect is

$$\Delta t_{\text{Sagnac}} \approx \frac{1}{c^2} \int_A^B (\boldsymbol{\omega} \times \mathbf{r}) \cdot d\mathbf{r} = \frac{2 \boldsymbol{\omega} A}{c^2}, \quad (12)$$

where  $A$  is the equatorial projection of the triangular area formed by the center of rotation and the endpoints of the light path.

If the receiver has velocity  $\mathbf{v}'_R$  relative to the ECEF frame, then  $\mathbf{v}_R = \mathbf{v}'_R + \boldsymbol{\omega} \times \mathbf{r}_R$  and the receiver velocity correction of Eq. (11) becomes the sum of two terms,

$$\Delta t_v = |\Delta \mathbf{r}| v'_R \cos \theta / c^2 + 2 \boldsymbol{\omega} A / c^2, \quad (13)$$

where  $\theta$  is the angle between the receiver velocity and the line joining the satellite and the receiver. The first term is the velocity correction in the ECEF frame and the second term is the Sagnac correction.

In the case of a receiver at rest on the Earth, an observer in the ECI frame sees that the receiver has moved due to the Earth’s rotation during the signal time of flight and applies a propagation time correction due to the additional path length. The correction applied by the inertial observer is the receiver “velocity correction.” However, an observer in the ECEF frame regards the receiver as stationary and applies the Sagnac correction. The term “Sagnac effect” is part of the vocabulary of only the observer in the rotating frame. While the interpretation of the correction is different in the two frames, the numerical value is the same in both frames.

## VI. THE GLOBAL POSITIONING SYSTEM

The GPS has served as a laboratory for doing relativity physics. The relativistic effects that are encountered in the GPS are not negligible. The consistent application of relativity to time and position measurements has been demonstrated by the operational precision of the system and by numerous experiments designed to test the individual effects over a wide range of conditions.

### *Relativistic Effects That Are Currently Modeled*

For measurements at the one-to-ten nanosecond level, there are three relativistic effects that must be considered [2].

First, there is the effect of time dilation. The velocity of a moving clock causes it to appear to run slow relative to a clock on the Earth. GPS satellites revolve around the Earth with an orbital period of 11.967 hours and a velocity of 3.874 km/s. Thus on account of its velocity, a GPS satellite clock appears to run slow by 7  $\mu$ s per day.

Second, there is the effect of the gravitational redshift, a frequency shift caused by the difference in gravitational potential. (The term “redshift” is generic, regardless of sign, but for a satellite clock the frequency shift is actually a “blueshift.”) The difference in gravitational potential between the altitude of the orbit and the surface of the Earth causes the satellite clock to run fast. At an altitude of 20 184 km, the clock runs fast by 45  $\mu$ s per day.

The net effect of time dilation and gravitational redshift is that the satellite clock runs fast by approximately 38  $\mu$ s per day when compared to a similar clock at rest on the geoid, including the effects of the Earth’s rotation and the gravitational potential at the Earth’s surface. This is an enormous rate difference for a clock that maintains time with a precision of a few nanoseconds over a day. To compensate for this large secular effect, the clock is given a fractional rate offset prior to launch of  $-4.465 \times 10^{-10}$  from its nominal frequency of exactly 10.23 MHz, so that when in orbit its average rate is the same as the rate of a clock on the ground. The actual frequency of the satellite clock prior to launch is thus 10.229 999 995 43 MHz.

Although GPS satellite orbits are nominally circular, there is always some residual eccentricity. The eccentricity causes the orbit to be slightly elliptical. Thus the velocity and gravitational potential vary slightly over one revolution and, although the principal secular effect is compensated by a rate offset, there remains a small residual variation that is proportional to the eccentricity. For example, with an orbital eccentricity of 0.02, there is a relativistic sinusoidal variation in the apparent clock time having an amplitude of 46 ns at the orbital period. This correction must be calculated and taken into account in the user’s receiver.

The third relativistic effect is the Sagnac effect. For a stationary terrestrial receiver on the geoid, the Sagnac correction can be as large as 133 ns (corresponding to a GPS signal propagation time of 86 ms and a velocity of 465 m/s at the equator in the ECI frame). This correction is also applied in the receiver.

### *Relativistic Effects That Are Not Currently Modeled*

Higher order effects that are not currently modeled in the GPS include the Earth oblateness contribution to the gravitational redshift, the tidal potentials of the Moon and Sun, and the effect of the gravitational potential on the speed of signal propagation. When satellite cross links are implemented in the future, the orbital eccentricity effect will have to be taken into account at both the transmitter and receiver.

The  $J_2$  oblateness component of the Earth’s gravitational potential makes a small contribution to the redshift. The coordinate time correction is

$$\Delta t_{\text{oblateness}} = \frac{1}{2} \frac{1}{c^2} \frac{GM}{a} J_2 \left( \frac{R}{a} \right)^2 \left[ \left( 1 - \frac{3}{2} \sin^2 i \right) \Delta \tau + \frac{\sin^2 i}{n} \sin(2n \Delta \tau) \right], \quad (14)$$

where  $a$  is the semimajor axis,  $i$  is the inclination, and  $n$  is the mean motion. For an inclination of  $55^\circ$  the amplitude is 24 ps.

By the Principle of Equivalence, the gravitational potentials of the Moon and Sun do not directly affect a clock on a satellite in orbit about the Earth because the Earth constitutes a freely falling frame of reference. Thus they appear as tidal effects. The tidal potential due to an external third body is approximately

$$U_{\text{tidal}} = \frac{1}{2} \frac{\partial^2 U_{\text{ext}}}{\partial x^i \partial x^j} x^i x^j \approx \frac{GM_{\text{ext}}}{r_E^3} (r_S - r_E)^2, \quad (15)$$

where  $r_S$  and  $r_E$  are the distances of the third body from the satellite and the Earth, and  $r_S - r_E \approx r \cos i \sin(n \Delta \tau)$ , where  $r$  is the orbital radius and  $i$  is the orbital inclination relative to the third body. Thus the correction to the coordinate time is

$$\Delta t_{\text{tidal}} \approx \frac{1}{2} \frac{1}{c^2} \frac{GM_{\text{ext}}}{r_E^3} \left( \frac{r}{r_E} \right)^2 \cos^2 i \left[ \Delta \tau - \frac{1}{2n} \sin(2n \Delta \tau) \right]. \quad (16)$$

For the Moon the secular drift rate is 15 ps per revolution and the amplitude of the periodic term is 1 ps. For the Sun the values are 7 ps per revolution and 0.5 ps, respectively. Although the Sun has a mass that is about 30 million times that of the Moon, it is about 400 times farther away. Since the tidal potential varies inversely as the cube of the distance, the tidal effect of the Sun is about one-half that of the Moon.

The relativistic signal propagation delay with respect to a clock on the geoid is

$$\Delta t_{\text{delay}} = -\frac{W_0 \rho}{c^3} + \frac{2GM}{c^3} \ln \left( \frac{R+r+\rho}{R+r-\rho} \right), \quad (17)$$

where  $R$  is the radius of the Earth,  $r$  is the radius of the orbit, and  $\rho$  is the path length. The first term is due to the change of scale and the second term is the gravitational time delay with respect to a clock at infinity.

Table 1: Relativistic effects on clocks and signals for satellites in Earth orbit.

| Satellite orbital properties        |                   |       |         |         |         |         |       |
|-------------------------------------|-------------------|-------|---------|---------|---------|---------|-------|
| Satellite                           |                   | ISS   | GLONASS | GPS     | Galileo | Molniya | GEO   |
| Semimajor axis                      | km                | 6766  | 25510   | 26561.8 | 29994   | 26562   | 42164 |
| Eccentricity                        |                   | 0.01  | 0.02    | 0.02    | 0.02    | 0.722   | 0.01  |
| Inclination                         | deg               | 51.6  | 64.8    | 55.0    | 56.0    | 63.4    | 0.1   |
| Argument of perigee                 | deg               | 270   | 270     | 270     | 270     | 250     | 270   |
| Apogee altitude                     | km                | 456   | 19642   | 20715   | 24216   | 39362   | 36208 |
| Perigee altitude                    | km                | 320   | 18622   | 19652   | 23016   | 1006    | 35364 |
| Ascending/descending node altitude  | km                | 387   | 19122   | 20173   | 23604   | 10507   | 35782 |
| Period of revolution                | s                 | 5539  | 40549   | 43082   | 51697   | 43083   | 86164 |
| Mean motion                         | rev/d             | 15.6  | 2.1     | 2.0     | 1.7     | 2.0     | 1.0   |
| Mean velocity                       | km/s              | 7.675 | 3.953   | 3.874   | 3.645   | 3.874   | 3.075 |
| Clock effects                       |                   |       |         |         |         |         |       |
| Secular time dilation               | μs/d              | -28.2 | -7.4    | -7.1    | -6.3    | -7.1    | -4.4  |
| Secular redshift                    | μs/d              | 3.5   | 45.1    | 45.7    | 47.3    | 45.7    | 51.0  |
| Net secular effect                  | μs/d              | -24.7 | 37.7    | 38.6    | 41.1    | 38.6    | 46.6  |
| Amplitude of eccentricity effect    | ns                | 12    | 45      | 46      | 49      | 1653    | 29    |
| Secular oblateness redshift         | ns/d              | 23.7  | 0.8     | 0.5     | 0.4     | 2.5     | -0.1  |
| Amplitude of oblateness effect      | ps                | 264   | 50      | 38      | 33      | 167     | 0     |
| Amplitude of lunar tidal effect     | ps                | 0.0   | 1.0     | 1.2     | 1.8     | 1.2     | 6.1   |
| Amplitude of solar tidal effect     | ps                | 0.0   | 0.5     | 0.5     | 0.8     | 0.5     | 2.7   |
| Signal propagation                  |                   |       |         |         |         |         |       |
| Maximum Sagnac effect               | ns                | 13    | 131     | 136     | 155     | 234     | 218   |
| Gravitational delay along radius    | ps                | 0.8   | -3.5    | -4.7    | -9.1    | -4.7    | -27.3 |
| Amplitude of periodic Doppler shift | 10 <sup>-12</sup> | 13.1  | 7.0     | 6.7     | 5.9     | 241.1   | 2.1   |

The analysis of electromagnetic signals used in time synchronization between satellite clocks and cross-link ranging involves three steps: (1) a relativistic transformation from the proper time reading of the clock at the transmitter to the coordinate time of transmission in the adopted coordinate system; (2) calculation of the coordinate time of signal propagation, including the relativistic gravitational delay; (3) a relativistic transformation from the coordinate time of reception in the adopted coordinate system to the proper time reading of the clock at the receiver. Therefore, the signal propagation time between two satellites as measured by the difference in readings of the satellite clocks is

$$\Delta\tau = \frac{|\Delta\mathbf{r}|}{c} + \frac{\Delta\mathbf{r} \cdot \mathbf{v}_R}{c^2} - \frac{2\mathbf{r}_R \cdot \mathbf{v}_R}{c^2} + \frac{2\mathbf{r}_T \cdot \mathbf{v}_T}{c^2} - \frac{W_0 |\Delta\mathbf{r}|}{c^3} + \frac{2GM}{c^3} \ln \left( \frac{r_T + r_R + \rho}{r_T + r_R - \rho} \right). \quad (18)$$

The first two terms are the geometric propagation time, the next two terms are the orbital eccentricity corrections, and the last two terms are the relativistic propagation delay. There may also be a nonrelativistic path delay due to the residual atmosphere.

For frequency measurements there is a relativistic Doppler correction. The Doppler shift is

$$\frac{\Delta f}{f_T} = \frac{1}{c} \left( 1 + \frac{1}{c} \mathbf{n} \cdot \mathbf{v}_T \right) \left[ \mathbf{n} \cdot (\mathbf{v}_T - \mathbf{v}_R) \right] + \frac{1}{c^2} (U_R - U_T) + \frac{1}{2} \frac{1}{c^2} (v_R^2 - v_T^2) \quad (19)$$

where  $\mathbf{n}$  is a unit normal along the path.

## VII. OTHER SATELLITES IN EARTH ORBIT

To illustrate their orders of magnitude, the relativistic effects on clocks and signal propagation for a variety of Earth orbiting satellites are compared in *Table 1*.

## VIII. TIME TRANSFER FROM MARS TO EARTH

The analysis of time transfer in the solar system must be carried out in a common coordinate system. A convenient coordinate system is one whose origin is at the solar system barycenter, in which the coordinate time is called Barycentric Coordinate Time (TCB). For time transfer between Mars and Earth, two transformations are required. The first transformation is from Terrestrial Time

(TT) to Barycentric Coordinate Time (TCB) and the second is from Barycentric Coordinate Time (TCB) to Mars Time (MT).

#### Barycentric Coordinate Time – Terrestrial Time

The elapsed coordinate time  $\Delta t$  in a barycentric coordinate system corresponding to the proper time  $\Delta \tau$  maintained by a clock, having an arbitrary position and velocity in this coordinate system, is

$$\Delta t = \int_{\tau_0}^{\tau} \left( 1 + \frac{1}{c^2} U(\mathbf{r}) + \frac{1}{2} \frac{1}{c^2} v^2 \right) d\tau, \quad (20)$$

where  $\mathbf{r}$  and  $\mathbf{v}$  are the barycentric position and velocity of the clock and  $U(\mathbf{r})$  is the gravitational potential of all the bodies in the solar system (including the Earth) evaluated at the clock. The integral depends on the position and velocity of the clock in the barycentric coordinate system. The coordinate time  $\Delta t$  is identified with TCB.

It is desirable to separate the clock-dependent part from the clock-independent part. In this approximation, the barycentric position and velocity of the clock may be expressed as  $\mathbf{r} = \mathbf{r}_E + \mathbf{R}$  and  $\mathbf{v} = \mathbf{v}_E + \dot{\mathbf{R}}$ , where  $\mathbf{r}_E$  and  $\mathbf{v}_E$  are the barycentric position and velocity of the Earth's center of mass, and  $\mathbf{R}$  and  $\dot{\mathbf{R}}$  are the geocentric position and velocity of the clock. The total potential is

$$U(\mathbf{r}) = U_E(\mathbf{r}) + U_{\text{ext}}(\mathbf{r}), \quad (21)$$

where  $U_E(\mathbf{r})$  is the Newtonian potential of the Earth and  $U_{\text{ext}}(\mathbf{r})$  is the external Newtonian potential of all of the solar system bodies apart from the Earth. The external potential may be expressed

$$U_{\text{ext}}(\mathbf{r}) \approx U_{\text{ext}}(\mathbf{r}_E) + \nabla U_{\text{ext}} \cdot \mathbf{R}. \quad (22)$$

Also,

$$v^2 = v_E^2 + 2 \dot{\mathbf{R}} \cdot \mathbf{v}_E + |\dot{\mathbf{R}}|^2 \quad (23)$$

where

$$\dot{\mathbf{R}} \cdot \mathbf{v}_E = \frac{d}{dt} (\mathbf{R} \cdot \mathbf{v}_E) - \mathbf{R} \cdot \mathbf{a}_E. \quad (24)$$

The Earth's acceleration in the barycentric coordinate system is

$$\mathbf{a}_E \equiv \frac{d\mathbf{v}_E}{dt} = \nabla U_{\text{ext}}. \quad (25)$$

Upon substitution of Eqs. (21) and (23) into Eq. (20), the gravitational acceleration terms  $\nabla U_{\text{ext}} \cdot \mathbf{R}$  and  $\mathbf{R} \cdot \mathbf{a}_E$  cancel out. Thus the elapsed coordinate time is [3, 4]

$$\begin{aligned} \Delta t \approx \Delta \tau &+ \frac{1}{c^2} \int_{t_0}^t \left( U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2} v_E^2 \right) dt \\ &+ \frac{1}{c^2} \int_{t_0}^t \left( U_E(\mathbf{R}) + \frac{1}{2} |\dot{\mathbf{R}}|^2 \right) dt + \frac{1}{c^2} \mathbf{R} \cdot \mathbf{v}_E \Big|_{t_0}^t. \end{aligned} \quad (26)$$

This equation is completely general, regardless of the position of the clock. The first term is the proper time measured by the clock. The second term is due to the combined redshift and time dilation effects at the geocenter with respect to the barycenter and is independent of the clock. The third term is the time difference between a clock at the geocenter and a clock at position  $\mathbf{R}$  with respect to the geocenter. The fourth term depends on the clock's velocity and position. In the limit of flat space-time, it represents the special relativity clock synchronization correction in the moving geocentric frame when observed from the barycentric frame. The cancellation of the two acceleration terms is a manifestation of the Principle of Equivalence for a freely falling frame of reference. That is, the Earth constitutes a freely falling frame in its orbit about the Sun.

The coordinate time scale of Terrestrial Time (TT) is equivalent to the proper time kept by a hypothetical clock on the geoid. This timescale is related to International Atomic Time (TAI) by the equation

$$TT = TAI + 32.184 \text{ s}. \quad (27)$$

The constant offset represents the difference between Ephemeris Time (an obsolete Newtonian timescale used for astronomical ephemerides which has been superseded by TT) and TAI at the defining epoch of TAI on January 1, 1958. For an actual clock at rest at an elevation  $h$  above the geoid where the local acceleration of gravity is  $g$ , the relation between TT and the proper time reading  $\Delta \tau$  of the clock is

$$TT = \Delta \tau' = (1 - g h / c^2) \Delta \tau. \quad (28)$$

The transformation from TT to Geocentric Coordinate Time (TCG) is

$$TCG - TT = (W_{0E} / c^2) \Delta T = L_G \Delta T, \quad (29)$$

where  $W_{0E}$  is the Earth's geopotential,  $L_G \equiv W_{0E} / c^2 = 6.969\,290\,134 \times 10^{-10} \approx 60.2 \text{ } \mu\text{s/d}$ , and  $\Delta T$  is the time elapsed since 1 January 1977 0 h TAI (JD 244 3144.5).

In Eq. (26) above, the first integral may be calculated by numerical integration [5] or it may be represented by an analytical formula [6]. It is expressed as the sum of a secular term  $L_C \Delta T$  and periodic terms  $P$ . For a clock on the geoid the second integral is simply  $W_{0E} \Delta t$ . Thus for a clock on the geoid

$$\begin{aligned} TCB &= TT + \frac{1}{c^2} \int_{t_0}^t \left( U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2} v_E^2 \right) dt \\ &+ L_G \Delta T + \frac{1}{c^2} \mathbf{R}_E \cdot \mathbf{v}_E \Big|_{t_0}^t \\ &= TT + L_C \Delta T + P + L_G \Delta T + \frac{1}{c^2} \mathbf{R}_E \cdot \mathbf{v}_E \Big|_{t_0}^t, \end{aligned} \quad (30)$$

where  $L_C = 1.480\,826\,867\,41 \times 10^{-8} \approx 1.28 \text{ ms/d}$ . For a

clock on the equator, the diurnal term has a maximum amplitude of 2.1  $\mu$ s. The leading terms in the evaluation of the integral are

$$\frac{1}{c^2} \int_{t_0}^t \left( U_{\text{ext}}(\mathbf{r}_E) + \frac{1}{2} v_E^2 \right) dt \approx \frac{3}{2} \frac{1}{c^2} \frac{GM_S}{a_E} \Delta T + \frac{2}{c^2} \sqrt{GM_S a_E} e_E \sin E_E, \quad (31)$$

where  $GM_S$  is the gravitational constant of the Sun, and where  $a_E$  and  $e_E$  are the Earth's orbital semimajor axis and eccentricity. The first term is an approximation to  $L_C \Delta T$ . The second term is the principal periodic term in  $P$ , which has amplitude of 1.7 ms.

#### Barycentric Coordinate Time – Mars Time

By an analogous derivation, one finds that the transformation from Barycentric Coordinate Time (TCB) to Mars Time (MT) is given by

$$\begin{aligned} \text{TCB} &= \text{MT} + \frac{1}{c^2} \int_{t_0}^t \left( U_{\text{ext}}(\mathbf{r}_M) + \frac{1}{2} v_M^2 \right) dt \\ &\quad + L_M \Delta T + \frac{1}{c^2} \mathbf{R}_M \cdot \mathbf{v}_M \Big|_{t_0}^t \\ &= \text{MT} + L_{CM} \Delta T + P + L_M \Delta T + \frac{1}{c^2} \mathbf{R}_M \cdot \mathbf{v}_M \Big|_{t_0}^t, \end{aligned} \quad (32)$$

where  $L_{CM} = 0.972 \times 10^{-8} \approx 0.84$  ms/d,  $L_M \equiv W_{0M} / c^2 = 1.403 \times 10^{-10} \approx 12.1$   $\mu$ s/d,  $W_{0M}$  is the areopotential ("geopotential" on Mars),  $P$  represents periodic terms, and  $\mathbf{R}_M$  is the areocentric position of the clock on the surface of Mars. The diurnal term has a maximum amplitude of 0.9  $\mu$ s. The principal periodic term in  $P$  has an amplitude of 11.4 ms.

#### Net Effects: Mars Time – Terrestrial Time

The difference in the readings of a clock on the surface of Mars and a clock on the surface of the Earth has both secular and periodic terms. The difference between Mars Time (MT) and Terrestrial Time (TT) is

$$\text{MT} - \text{TT} = (\text{TCB} - \text{TT}) - (\text{TCB} - \text{MT}). \quad (33)$$

The net secular drift rate is  $(1.28 \text{ ms/d} + 0.06 \text{ ms/d}) - (0.84 \text{ ms/d} + 0.01 \text{ ms/d}) = 0.49 \text{ ms/d}$ . The amplitudes of the periodic variations are (a) 1.7 ms at the Earth orbital period (365.2422 d) and (b) 11.4 ms at the Mars orbital period (687 d). Therefore, in the transfer of time between a clock on Mars and a clock on the Earth, there are secular and periodic effects that are on the order of one to ten milliseconds. For a navigation ranging system referenced to a clock on Mars and an ephemeris referenced to clocks on Earth, the radial position error could be as much as 3000 km if the relativistic effects were not modeled.

## IX. TIME TRANSFER FROM THE MOON TO EARTH

For time transfer from the surface of the Moon to the surface of the Earth, the procedure is similar but the relative magnitudes of the terms are different. A convenient coordinate system is one whose origin is at the center of the Earth. (The motion of the Earth's center about the center of mass of the Earth-Moon system will be neglected.) The difference between Geocentric Coordinate Time (TCG) and Terrestrial Time (TT) is again given by

$$\text{TCG} - \text{TT} = L_G \Delta T. \quad (34)$$

But TCG is related to Lunar Time (LT), the proper time measured by clocks on the Moon's surface, by the equation

$$\text{TCG} = \text{LT} + L_{Cm} \Delta T + P + L_m \Delta T + \frac{1}{c^2} \mathbf{R}_m \cdot \mathbf{v}_m \Big|_{t_0}^t, \quad (35)$$

where  $L_{Cm} = 1.731 \times 10^{-11} = 1.5$   $\mu$ s/d, the amplitude of the periodic term is 0.48  $\mu$ s, and  $L_m = 3.141 \times 10^{-11} = 2.7$   $\mu$ s/d. The difference between Lunar Time (LT) and Terrestrial Time (TT) is

$$\text{LT} - \text{TT} = (\text{TCG} - \text{TT}) - (\text{TCG} - \text{LT}). \quad (36)$$

The net drift rate is  $60.2 \text{ } \mu\text{s/d} - (1.5 \text{ } \mu\text{s/d} + 2.7 \text{ } \mu\text{s/d}) = 56.0 \text{ } \mu\text{s/d}$ .

## X. CONCLUSION

Time transfer between clocks operating on Earth and clocks at Mars or on the Moon will become an essential activity for future space missions. The analysis of this paper has shown that relativistic effects for solar system time transfer, as in the GPS, are not negligible. The relativistic effects at Mars comprise a secular rate difference of about 0.49 ms/d and periodic variations with amplitudes of 1.7 ms and 11.4 ms relative to Earth-based clocks. For the Moon, the secular effect is an order of magnitude less and the periodic effect is four orders less.

## REFERENCES

- [1] R. A. Nelson, "Relativistic Effects," in *Handbook on Satellite Time and Frequency Transfer and Dissemination* (ITU, Geneva, 2007), in press.
- [2] N. Ashby and J. J. Spilker, Jr., "Introduction to Relativistic Effects in the Global Positioning System," in B. W. Parkinson and J. J. Spilker, Jr. (editors), *Global Positioning System: Theory and Applications* (AIAA, Washington, DC, 1996), Vol. I, pp. 623 – 697.
- [3] J. B. Thomas, "Reformulation of the Relativistic Conversion Between Coordinate Time and Atomic Time," *Ap. J.* **80**, 405 – 411 (1975).
- [4] T. D. Moyer, "Transformation from Proper Time on Earth to Coordinate Time in Solar System Barycentric Space-Time Frame of Reference," *Celestial Mech.* **23**, 33 – 68 (1981).
- [5] A. W. Irwin and T. Fukushima, "A Numerical Time Ephemeris of the Earth," *Astron. Astrophys.* **348**, 642 – 652 (1999).
- [6] L. Fairhead, P. Bretagnon, and J.-F. Lestrade, "The Time Transformation TB – TT: An Analytical Formula and Related Problem of Convention," in *The Earth's Rotation and Reference Frames for Geodesy and Geophysics*, edited by A. K. Babcock and G. A. Wilkins (Kluwer, Dordrecht, 1988), pp. 419 – 426.